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# SIMPLE DEMONSTRATION OF TIME-REVERSAL INVARIANCE IN CLASSICAL MECHANICS\*

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The principle of time-reversal invariance can be simply and vividly demonstrated by the successive elastic collisions of two steel ball bearings of different mass, suspended as the bobs of two equal-length pendulums. (I discovered this accidentally while playing with an apparatus I designed to demonstrate something entirely different.)

Time-reversal invariance<sup>1</sup> can be described as follows. A system of particles in some initial state  $[i]$  undergoes a dynamic process that carries it at some later time to a final state  $[f]$ . The state  $[i]$  is given in terms of the positions and momenta of all the particles at the initial time. Similarly for the final state  $[f]$ . The dynamic process involves the interactions of the particles. Suppose we take a moving picture of this process  $[i] \rightarrow [f]$ . Project this on the screen so that we see a visual representation of the process  $[i] \rightarrow [f]$ . At the instant that the picture reaches  $[f]$ , reverse the direction of the motion of the film through the projector. On the screen we then see the particles at the same positions as in  $[f]$ , but with all momenta reversed. Call this "the time-reversed final state," designated by  $[Tf]$ . As the projector plays backwards we see the dynamic process apparently unfold in the backwards direction, carrying the time-reversed

final state to a similarly time-reversed initial state:

$$[Tf] \rightarrow [Ti] . \quad (1)$$

Now suppose that we prepare the actual system of particles so as to be in the state  $[Tf]$  that would have the same appearance as  $[f]$ , but with all momenta reversed. Then we let the interactions of the particles take over. Will the subsequent dynamic process carry the system to  $[Ti]$ ? If so, the appearance of the process will be exactly as that of the reversed-projector motion that gave us Eq. (1) on the screen. If Eq. (1) holds for the actual process we say that the interaction is invariant under the operation of time reversal, or satisfies time-reversal invariance.

Newton's laws guarantee that a process that involves elastic collisions between a system of particles satisfies time-reversal invariance. However, this invariance is difficult to demonstrate because it is practically impossible to prepare the state  $[Tf]$  if there are many particles. Therefore we confine ourselves, at first, to two particles.

The collisions between the two particles must be elastic. Otherwise there are actually many particles involved, the energy gets "degraded" into heat, and time-reversal invariance cannot be demonstrated because of the practical impossibility of preparing the state  $[Tf]$ .

An unsatisfactory demonstration is to observe the elastic collision of two equal masses. The time-reversed final state  $[Tf]$  is easy to prepare, but it is too closely related to the initial state. In fact  $[Tf]$  is simply equal to the initial state  $[i]$  as seen from a different observation position, or as seen from the same position

with the entire apparatus rotated by  $180^\circ$  about a vertical axis. Then the time-reversed process is just a trivial repetition of the original process from a different viewpoint. To obtain a non-trivial demonstration of time-reversal invariance we must consider the elastic collision of two unequal masses.

The problem of preparing the time-reversed state  $[Tf]$  is neatly solved by attaching the unequal masses to strings so as to have two simple one-dimensional pendulums of equal length. The two unequal masses are hung so that at equilibrium they barely touch. They are then displaced in opposite directions by arbitrary unequal amounts to initial positions from which they are simultaneously released from rest. (A particularly simple configuration is to displace only one mass from equilibrium.) Because of the remarkable property of the simple pendulum, that its period is independent of the mass and of the amplitude (for sufficiently small amplitudes), the two masses reach their equilibrium point simultaneously. Their velocities and momenta are unequal. Call this the state  $[i]$ . Then they undergo an elastic collision. (Steel ball bearings are sufficiently elastic.) After the collision the two ball bearings have nontrivial new momenta, and are in state  $[f]$ . Next the pendulum motion takes over. The two particles rise to new positions, come to rest simultaneously, reverse their motions, and arrive simultaneously back at the equilibrium point. Just before they collide for the second time they are at the same positions as they were in  $[f]$ , but with reversed momenta. They are thus in the time-reversed state  $[Tf]$ , and all because of that remarkable property of a simple pendulum! Therefore the second

collision should carry them to the time-reversed initial state  $[Ti]$ , provided time-reversal invariance holds for the collision process. After that the pendulum motion carries them back to the original positions from which they were originally released from rest. They arrive there simultaneously. Then the whole process repeats.

It works! Time-reversal invariance becomes visible! I start with a commercial pendulum toy<sup>2</sup> consisting of five equal-mass ball bearings each cemented to two threads which are hung from two parallel rails so that the balls are constrained to swing in a plane. I hang four of the balls out of the way. I take a thread of the correct length, cement its center with Epoxy cement to a smaller ball bearing, attach two bent-wire hooks to the thread ends, and suspend the hooks from the rails so that the small ball bearing is at the same height as the remaining large ball, and barely in contact with it at their equilibrium positions.<sup>3,4</sup> I tape the hooks to the rails so they won't slide.

Perhaps the most spectacular demonstration is where one of the balls is initially in its equilibrium position and the other is displaced and released from rest. After the first collision and the time reversal of  $[f]$  by the pendulum action the two balls are both in motion approaching each other with different velocities. Then a psychologically most unexpected thing happens. After the second collision one of the balls is at rest! Because of the asymmetry—different size balls, different velocities—this is startling. One is not accustomed to seeing time reversal in action! It is fascinating to watch the whole complicated-appearing process repeat itself every two collision. The ear readily records

the sharp "clicks" of the collisions and one sees the process repeated every two clicks. Because of inelasticity the whole system gradually comes to rest, but it is easy to see time reversal in action for half a dozen complete cycles of two clicks each.

Next we come to something new which was extremely startling to me at first, and took me days to understand. I decided to see what would happen with three balls. I took down one of the balls I had previously hung out of the way, so that I had now three balls altogether, two equal balls from the original toy and the one smaller ball I had added. All three balls hung at equilibrium barely in contact, the two large ones contacting each other and the smaller one contacting one of the large ones. I pulled back the small ball and let it go. I expected that after one click I would see some complicated pattern, but that, by analogy with my experience with two balls, I would observe the return of the system after two clicks to its original configuration; then the entire process would repeat every two clicks. Instead, to my amazement, it took four clicks for the process to repeat! (Each click is a three-body collision. The fact that all three bodies do not collide precisely simultaneously cannot be distinguished by ear or eye.) By slight adjustment of the position of the hooks supporting the small ball, I later found that if I set the small ball so that it was separated by a millimeter from the nearest large ball I could get the process to repeat every two clicks as originally expected. But by slightly moving the small ball so that it was not only in contact but slightly "leaning" on the nearest large ball I found I could get the process to repeat every four clicks, or six clicks, or eight or ten clicks!

What is happening here? Experimentally it is striking and unambiguous to every observer. The configurations of the three moving balls remain complicated until suddenly, after say four or six clicks (always an even number!) the original simple configuration miraculously recurs. Then the whole process repeats, always with the same even number of clicks per repetition cycle. To change the number of clicks per repetition one moves the hooks a tiny amount. The only cycle that is easily reproduced at will is that for two clicks. Before I give my explanation I invite the reader to cover up the remainder of this note and think about the problem.

My explanation is that: Each click is actually two unresolved collisions, since it is impossible for all three balls to collide precisely simultaneously. In the first click the small ball, call it  $s$ , first hits the nearest large ball,  $L_1$ , then the large ball  $L_1$  hits the adjoining large ball  $L_2$ . After that,  $L_1$  is at rest (because its mass is equal to that of  $L_2$ ) and  $L_2$  is in motion. Next the pendulum motion takes over, so that  $L_2$  and  $s$  swing out, stop, reverse, and return to initiate the second click—the second three-body collision. It is now absolutely crucial which collision takes place first, that between  $L_2$  (moving) and  $L_1$  (at rest) or, instead, that between  $s$  (moving) and  $L_1$  (at rest). If  $s$  has been hung so it does not quite touch  $L_1$  at equilibrium then  $s$  will reach the collision a little late. In that case the  $L_2$ - $L_1$  collision occurs first. That interchanges the momentum of  $L_2$  and  $L_1$ , bringing  $L_2$  to rest. Then  $L_1$  and  $s$  have the time-reversed motion they acquired after the first collision of the first click. They now

collide, bringing  $L_1$  to rest, and  $s$  to its time-reversed initial motion. Now  $L_2$  and  $L_1$  are at rest. The small ball  $s$  swings out to its initial position, and the entire process is ready to repeat itself. Thus the entire process repeats itself every 2 clicks, which means, actually, every four collisions. The second click is the time reverse of the first.

Now consider what happens if  $s$  is hung so as to lean against  $L_1$  at equilibrium. The two collisions of the first click proceed in the same order as described above, leaving  $L_1$  at rest and  $L_2$  and  $s$  in motion. They swing out, reverse, and return for the second click. This time  $s$  gets back a little early, colliding with the resting  $L_1$ . That is the same kind of collision  $s$  made during the first click, so that  $s$  recoils and swings out the same way it did the first time. Similarly,  $L_1$  now has the same motion it had after the first collision of the first click, and it goes on to collide with  $L_2$ . But this time  $L_2$  is not at rest! In fact it has equal and opposite velocity to  $L_1$ . They collide in the second collision of the second click, interchanging their momenta. Thus after the second click,  $s$  and  $L_1$  are moving in the same direction, and  $L_2$  is moving in the opposite direction with momentum magnitude equal to that of  $L_1$ . The system has not returned after two clicks to its original configuration! What happens now? That depends on the order of the two collisions in the third click! It is possible for the third click to be the time reverse of the second click. That will be the case if  $L_2$  and  $L_1$  collide first. (They collided last in the second click, so must collide first to obtain the time-reversed click.) If  $L_2$  and  $L_1$  also collide first in the fourth

click, then the entire first two clicks will have been time reversed in the third and fourth clicks, and the entire process repeats every four clicks (eight collisions). However if, in the third click,  $s$  and  $L_1$  collide first, then the time-reversal process (the running backwards of the movie projector!) has not yet started, and it will be at least six clicks before the entire process repeats. Indeed, if  $s$  and  $L_1$  continue to collide first, the time-reversed process will never occur! Even if, during some click,  $L_2$  and  $L_1$  collide first, there is no guarantee that the entire process up to that time will now unfold in the time-reversed order. Sometimes a repeating subcycle of two or four or six clicks is observed, where the original initial condition is never recovered, but a new configuration becomes the effective initial condition. Such a subcycle must consist of a certain sequence of collisions followed by the time-reversed sequence. Such subcycles must therefore have an even number of clicks. And they do!

It would be nice to take a movie of a six-click cycle, make a time-reversed copy of the film so the clicks occur in order 6, 5, 4, 3, 2, 1, and then project the two films simultaneously in slow motion so that the time-reversed 6 goes with forwards 1, reversed 5 with forwards 2, and reversed 4 with forwards 3. If these two projections looked the same, that would verify that the last half of the cycle is precisely the time reverse of the first half.

Footnote and References

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1. For a beautifully written discussion of the role of time reversal in classical physics and in elementary particle physics, see Robert G. Sachs, Science 176, 587 (12 May 1972).

2. Pendulation, North Pacific Products Co., Bend, Oregon.  
Found in many toy and gift shops.

3. Actually, this was done for me by Hagop Hagopian and Ronald Coverson. The apparatus they built for me had a sequence of gradually decreasing suspended masses in contact, and was intended to demonstrate impedance matching in the transfer of energy between a large and a small ball. While playing with this I noticed the time-reversal effect, which excited me more than the original demonstration.

4. While writing this note it occurs to me that a quicker method of constructing the apparatus might be to start with the toy (Ref.2) and simply cement two of the masses together. However, this apparatus would not have the flexibility that led to my discovery of the complex time-reversal phenomena discussed in the last part of this note.

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